Generalizations of the Ostrowski’s inequality

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Abstract
Using the Taylor-Langrange formula as well as a generalization of this one, we give some generalizations of the integral midpoint inequality as well as the Ostrowski inequality for n-time differentiable mappings. A new sharp generalized weighted Ostrowski type inequality is given.

Keywords: Generalized Ostrowski’s inequality, generalized midpoint inequality, Taylor’s formula.

1. Introduction
Integral inequalities have been used extensively in most subjects involving mathematical analysis.

They are particularly useful for approximation theory and numerical analysis in which estimates of approximation errors are involved.

In 1938, Ostrowski (see for example [2], [5], [6]) proved the following integral inequality:

\[
\left| f(x) - \frac{1}{b-a} \int_a^b f(t) \, dt \right| \leq \left[ \frac{1}{4} + \left( \frac{x-a-b}{(b-a)^2} \right)^2 \right] (b-a) \| f' \| \infty ,
\]

\[ (1.1) \]

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for all \( x \in [a, b] \), where the mapping \( f : [a, b] \to \mathbb{R} \) is differentiable on \((a, b)\), and \( f'\) is bounded on \((a, b)\), which means \( \|f'\|_\infty := \sup_{x \in (a, b)} |f'(t)| < \infty \). The constant \( \frac{1}{4} \) is sharp in the sense that it can not be replaced by a smaller one.

On the Numerical Integration is also important, the following inequality (see for example \([3], [5], [6]\)) well known as the integral midpoint inequality:

\[
\left| \frac{b-a}{b-a} \int_a^b f(t) dt - f \left( \frac{a+b}{2} \right) \right| \leq \frac{(b-a)^2}{24} \|f''\|_\infty ,
\]

where the mapping \( f : [a, b] \to \mathbb{R} \) is twice differentiable on \((a, b)\), and \( f'' \) is bounded on \((a, b)\).

For some applications of Ostrowski’s inequality to some special means and some numerical quadrature rules, we refer to the recent paper \([4]\) by S. S. Dragomir and S. Wang.

Dragomir, Cerone and Sofa \([3]\), using the celebrated Hermite-Hadamard inequality proved the following inequality, which is better than (1.2):

\[
\frac{\gamma}{24} \frac{(b-a)^2}{24} \leq \frac{1}{b-a} \int_a^b f(t) dt - f \left( \frac{a+b}{2} \right) \leq \frac{\Gamma}{24} \frac{(b-a)^2}{24} ,
\]

where \( \gamma := \inf_{x \in (a, b)} f''(x) \), \( \Gamma := \sup_{x \in (a, b)} f''(x) \).

In \([2]\), Cerone, Dragomir and Roumeliotis considered another midpoint type inequality, as follows:

Let \( f \) be as above. Moreover, if \( f'' \in L_1(a, b) \), then we have

\[
\left| \frac{1}{b-a} \int_a^b f(t) dt - f \left( \frac{a+b}{2} \right) \right| \leq \frac{b-a}{8} \|f''\|_1 ,
\]

where \( \|f''\|_1 = \int_a^b |f''(x)| dx \) i.e \( \| \cdot \|_1 \) is the norm of \( L_1(a, b) \).

G. V. Milovanovic and J. E. Pecaric (see for example \([6, p. 468]\)) proved the following generalization of Ostrowski’s inequality:

Let \( f : [a, b] \to \mathbb{R} \) be a \( n \)-times differentiable function, \( n \geq 1 \), and such that \( \|f^{(n)}\|_\infty < \infty \).
Then
\[
\left| \frac{f(x)}{n} + \sum_{k=1}^{n-1} \frac{(n-k)(x-a)^k f^{(k-1)}(a) - (x-b)^k f^{(k-1)}(b)}{k!n} \right| \leq \frac{\|f^{(n)}\|_{\infty}}{n(n+1)!} \cdot \frac{1}{b-a} \int_a^b f(x) \, dx
\]
(1.5)

for all \( x \in [a, b] \).

In this paper, by using the Taylor’s formula with Langrange’s form of remainder, as well as a generalization of the Taylor’s formula, we present a general weighted analytical integral Ostrowski type inequality involving \( n \)-time differentiable mappings. Also, based on this inequality, we prove a sharp extension of the classical Ostrowski inequality (1.1) and some new sharp generalizations of the inequalities (1.1), (1.2), (1.3) for \( n \)-time differentiable mappings are obtained.

Moreover, we give a generalized midpoint-Gruess type inequality for \( n \)-time differentiable mappings, which in special case yields an improvement of the inequality (1.4).

All these inequalities can be used in the theory of Numerical Integration and generally in Mathematical Analysis.

The paper is organized as follows: In section 2, we prove the general weighted inequality (Theorem 2) and all our results over Ostrowski’s inequality. The section 3 is devoted to obtain the generalized midpoint inequalities.

2. Generalizations of the Ostrowski inequality

Let us denote \( R_n(f; a, b) := f(b) - \sum_{i=0}^{n} \frac{(b-a)^i}{i!} f^{(i)}(a) \) the Taylor’s remainder. Moreover, for any \( c, d \in \mathbb{R} \), we denote by \( ([c, d]) \) the open interval \( \min\{c, d\}, \max\{c, d\} \), and by \( \left[ [c, d] \right) \) the closed \( \min\{c, d\}, \max\{c, d\} \). For the proof of the following Theorem 2, we need a generalization of the Taylor’s formula [1, p. 113]:

**Theorem 1.** If \( f^{(n)}, g^{(n)} \) are continuous on \( ([a, b]) \), \( f^{(n+1)}, g^{(n+1)} \) exist on \( ([a, b]) \), and \( g^{(n+1)}(x) \neq 0 \) for any \( x \in ([a, b]) \), then there is a number \( \xi \in ([a, b]) \), such that

\[
\frac{R_n(f; a, b)}{R_n(g; a, b)} = \frac{f^{(n+1)}(\xi)}{g^{(n+1)}(\xi)}.
\]
Theorem 2. Let \( I \subset \mathbb{R} \) be a closed interval. Let \( a, b, t \) be any numbers in \( I \) such that \( b > a \). Let \( f, g \in C^n(1) \). Suppose that \( f, g \) are differentiable of order \( n + 1 \) on the interior \( I \) of \( I \) such that \( g^{(n+1)} > 0 \) or \( g^{(n+1)} < 0 \) on \( I \) and \( \frac{f^{(n+1)}}{g^{(n+1)}} \) is bounded on \( I \). Let \( w : [a, b] \to \mathbb{R} \) be an integrable and positive valued mapping. Then for all \( x \in I - (a, b) \),

\[
R_n(f; x, t) = \frac{f^{(n+1)}(x)}{g^{(n+1)}(x)},
\]

or

\[
\inf_{t \in \{x, t\}} \frac{f^{(n+1)}(t)}{g^{(n+1)}(t)} \leq \frac{R_n(f; x, t)}{R_n(g; x, t)} \leq \sup_{t \in \{x, t\}} \frac{f^{(n+1)}(t)}{g^{(n+1)}(t)},
\]

and since obviously \( m \leq \inf_{y \in \{x, t\}} \frac{f^{(n+1)}(y)}{g^{(n+1)}(y)} \leq \sup_{y \in \{x, t\}} \frac{f^{(n+1)}(y)}{g^{(n+1)}(y)} \leq M \),

we get that for all \( t \in (a, x) \cup (x, b) \),

\[
m \leq \frac{R_n(f; x, t)}{R_n(g; x, t)} \leq M. \tag{2.2}
\]

Applying the Taylor-Lagrange formula to \( g \) about \( x \), we have that for some \( \sigma \in \{(t, x)\} \),

\[
R_n(g; x, t) = \frac{(t - x)^{n+1}}{(n + 1)!} g^{(n+1)}(\sigma).
\tag{2.3}
\]

We distinguish two cases.

**First case**: \( n \) is odd. Then from (2.3), and the assumptions of this theorem, for all \( x \in I - \{t\} \), follows

\[
\text{sign}(g^{(n+1)}) R_n(g; x, t) w(t) > 0. \tag{2.4}
\]
Combining (2.2) with (2.4) we get

\[ m \, \text{sign}(g^{(n+1)}) R_n(g; x, t) w(t) \leq \text{sign}(g^{(n+1)}) R_n(f; x, t) w(t) \leq M \text{sign}(g^{(n+1)}) R_n(g; x, t) w(t), \]  

(2.5)

for all \( x \in I - \{t\} \), and since (2.5) is true at \( t = x \), we have that (2.5) is true for all \( t \in I \). Now, integrating (2.5) over \([a, b]\), we obtain

\[ m \, \text{sign}(g^{(n+1)}) \int_a^b w(t) R_n(g; x, t) dt \leq \text{sign}(g^{(n+1)}) \int_a^b w(t) R_n(f; x, t) dt \leq M \text{sign}(g^{(n+1)}) \int_a^b w(t) R_n(g; x, t) dt. \]  

(2.6)

Integrating on both sides of (2.4) over \([a, b]\), we obtain

\[ \text{sign}(g^{(n+1)}) \int_a^b w(t) R_n(g; x, t) dt > 0. \]  

Consequently, from (2.6) we get

\[ m \leq \frac{\int_a^b w(t) R_n(f; x, t) dt}{\int_a^b w(t) R_n(g; x, t) dt} \leq M, \]

or

\[ m \leq \frac{\int_a^b \left( w(t) f(t) - \sum_{i=0}^n \frac{(t-x)^i}{i!} f^{(i)}(x) dt \right)}{\int_a^b \left( w(t) g(t) - \sum_{i=0}^n \frac{(t-x)^i}{i!} g^{(i)}(x) dt \right)} \leq M. \]  

(2.7)

From (2.7) we get immediately (2.1).

**SECOND CASE:** \( n \) is even. Then from (2.3), and the assumptions of this theorem we have that for all \( x \in I - (a, b) \), and all \( t \in (a, b) \),

\[ \text{sign}(t-x) \text{sign}(g^{(n+1)}) R_n(g; x, t) w(t) > 0. \]  

(2.8)

Combining (2.2) with (2.8), we obtain

\[ m \text{sign}(t-x) \text{sign}(g^{(n+1)}) R_n(g; x, t) w(t) \]

\[ \leq \text{sign}(t-x) \text{sign}(g^{(n+1)}) R_n(f; x, t) w(t) \]

\[ \leq M \text{sign}(t-x) \text{sign}(g^{(n+1)}) R_n(g; x, t) w(t). \]  

(2.9)
Now from (2.9), working similarly as in the first case, we obtain, that for all \( x \in I = (a, b) \) the estimation (2.1) holds.

**Theorem 3.** If \( f \in C^n([a, b]) \), \( f^{(n+1)} \) is bounded on \((a, b)\), then for any \( x \in (a, b) \), it follows

\[
\int_a^b f(t)dt - \sum_{i=0}^n \frac{(b-x)^i - (a-x)^i + 1}{(i+1)!} f^{(i)}(x) \leq \frac{(b-a)^{n+2} + (x-a)^{n+2}}{(n+2)!} \sup_{t \in [a,b]} |f^{(n+1)}(t)|, \tag{2.10}
\]

Specially, if \( n \) is odd, then the inequality (2.10) is sharp in the sense that the constant \( \frac{1}{(n+2)!} \) cannot be replaced by a smaller one.

**Proof.** Let \( t \) be any number in \([a, b]\) such that \( t \neq x \). Applying the Taylor-Lagrange formula to \( f \) about \( x \) we obtain

\[
f(t) - \sum_{i=0}^n \frac{(t-x)^i}{i!} f^{(i)}(x) = \frac{(t-x)^{n+1}}{(n+1)!} f^{(n+1)}(\xi),
\]

for some \( \xi \in (\min\{t, x\}, \max\{t, x\}) \subset (a, b) \). From this, we conclude, that for any \( t \in (a, b) \),

\[
\left| f(t) - \sum_{i=0}^n \frac{(t-x)^i}{i!} f^{(i)}(x) \right| \leq \frac{|(t-x)|^{n+1}}{(n+1)!} \sup_{t \in [a,b]} |f^{(n+1)}(t)|. \tag{2.11}
\]

By integrating on both sides of (2.11) over \([a, b]\), we get

\[
\int_a^b \left| f(t) - \sum_{i=0}^n \frac{(t-x)^i}{i!} f^{(i)}(x) \right| dt \leq \frac{\sup_{t \in [a,b]} |f^{(n+1)}(t)|}{(n+1)!} \int_a^b |(t-x)|^{n+1} dt,
\]

and hence

\[
\left| \int_a^b f(t)dt - \sum_{i=0}^n \frac{f^{(i)}(x)}{i!} \int_a^b (t-x)^i dt \right| \leq \frac{\sup_{t \in [a,b]} |f^{(n+1)}(t)|}{(n+1)!} \left( \int_a^x (x-t)^{n+1} dt + \int_x^b (t-x)^{n+1} dt \right).
\]

From this, we get immediately the estimation (2.10). For the sharpness of the inequality (2.10) by \( n \) odd, let us choose \( f(x) = x^{n+1} \).
Then we have
\[
\frac{(b - x)^{n+2} + (x - a)^{n+2}}{(n + 2)!} \sup_{t \in [a,b]} |f^{(n+1)}(t)|
\]
\[
= \frac{(b - x)^{n+2} + (x - a)^{n+2}}{(n + 2)!}.
\] (2.12)

On the other hand, we have
\[
\int_a^b f(t)\,dt - \sum_{i=0}^n \frac{(b - x)^{i+1} - (a - x)^{i+1}}{(i + 1)!} f^{(i)}(x)
\]
\[
= \frac{b^{n+2} - a^{n+2}}{n + 2} - \sum_{i=0}^n \frac{(b - x)^{i+1} - (a - x)^{i+1}}{(i + 1)!} n \cdots (n+2-i)x^{n+1-i}
\]
\[
= \frac{1}{n + 2} \left( b^{n+2} - \sum_{i=0}^n \frac{(b - x)^{i+1}}{(i + 1)!} x^{n+1-i} \right) - \frac{1}{n + 2} \left( a^{n+2} - \sum_{i=0}^n \frac{(a - x)^{i+1}}{(i + 1)!} x^{n+1-i} \right)
\]
\[
= \frac{1}{n + 2} \left( b^{n+2} - \sum_{j=1}^{n+1} \binom{n+2}{j} \frac{(b - x)^j}{j!} x^{n+2-j} \right) - \frac{1}{n + 2} \left( a^{n+2} - \sum_{j=1}^{n+1} \binom{n+2}{j} \frac{(a - x)^j}{j!} x^{n+2-j} \right)
\]
\[
= \frac{1}{n + 2} \left( (b^{n+2} - ((b - x + x)^{n+2} - x^{n+2} - (b - x)^{n+2})) - (a^{n+2} - ((a - x + x)^{n+2} - x^{n+2} - (a - x)^{n+2})) \right)
\]
\[
= \frac{(b - x)^{n+2} + (x - a)^{n+2}}{(n + 2)!}.
\] (2.13)

From (2.12) and (2.13), follows
\[
\left| \int_a^b f(t)\,dt - \sum_{i=0}^n f^{(i)}(x) \frac{(b - x)^{i+1} - (a - x)^{i+1}}{(i + 1)!} \right|
\]
\[
= \frac{(b - x)^{n+2} + (x - a)^{n+2}}{(n + 2)!} \sup_{t \in [a,b]} |f^{(n+1)}(t)|.
\] (2.14)

From (2.14), we conclude, that the constant \( \frac{1}{(n + 2)!} \) is the best possible in (2.10).
\[\square\]
Remark 1. Putting in (2.10) \( n = 0 \), we get
\[
\left| f(t) - \frac{1}{b-a} \int_a^b f(t)dx \right| \leq \frac{(t-a)^2 + (b-t)^2}{2(b-a)} \| f' \|_{\infty}.
\]
A simple calculation yields
\[
\frac{(t-a)^2 + (b-t)^2}{2(b-a)} = \left[ \frac{1}{4} + \frac{(t - \frac{a+b}{2})^2}{(b-a)^2} \right] (b-a).
\]
Combining both relations, the Ostrowski’s inequality (1.1) is obtained.

If \( n \) is odd, then the inequality (2.10) can be replaced by a better one:

**Theorem 4.** Let \( n \) be an odd positive integer. Let \( f \) be a mapping as in Theorem 3. Then
\[
\gamma_{n+1} \frac{(b-x)^{n+2} + (x-a)^{n+2}}{(n+2)!} \leq \int_a^b f(t)dt - \sum_{i=0}^{n} f^{(i)}(x) \frac{(b-x)^{i+1} - (a-x)^{i+1}}{(i+1)!} \leq \Gamma_{n+1} \frac{(b-x)^{n+2} + (x-a)^{n+2}}{(n+2)!} \quad (2.15)
\]
where \( \gamma_{n+1} := \inf_{x \in (a,b)} f^{(n+1)}(x) \), \( \Gamma_{n+1} := \sup_{x \in (a,b)} f^{(n+1)}(x) \). The inequality (2.15) is sharp.

**Proof.** We apply Theorem 2 by choosing \( I = [a, b] \), \( w(t) = 1 \), \( g(t) := t^{n+1} \). Further, by applying the Taylor-Langrange formula to \( g \) about \( x \), we get
\[
g(t) = \sum_{i=0}^{n} \frac{(t-x)^i}{i!} g^{(i)}(x) = (t-x)^{n+1}. \quad (2.16)
\]
Putting (2.16) in (2.1) we obtain (2.15). Choosing \( f(x) := x^{n+1} \), the equality in (2.15) holds, and hence the sharpness of the inequality is proved.

In the following theorem an extension of Ostrowski’s inequality (1.1) is given.

**Theorem 5.** Let \( f \) be a differentiable mapping with bounded \( f' \) on the interior \( I \) of an interval \( I \subset \mathbb{R} \). Then for all \( a, b \in I \) with \( b > a \) and all \( x \in I - (a, b) \),
holds
\[ |f(x) - \frac{1}{b-a} \int_a^b f(t) \, dt| \leq \frac{|b + a - 2x|}{2} \sup_{x \in I} |f'(x)|. \quad (2.17) \]

The inequality (2.17) is sharp in the sense that the constant \( \frac{1}{2} \) cannot be replaced by a smaller one.

**Proof.** Applying Theorem 2 for \( n = 0 \), \( w(x) = 1 \), \( g(x) = x \), we obtain
\[ \inf_{x \in I} f'(x) \leq \int_a^b f(t) \, dt - (b - a) f(x) \leq \sup_{x \in I} f'(x). \quad (2.18) \]
From this easily, we get (2.17). The equality in (2.17) is verified by any first degree polynomial, and hence the sharpness of inequality is proved. \( \square \)

**Remark 2.** For all \( x \in \mathbb{R} \), a simple calculation yields
\[ \frac{|b + a - 2x|}{2} \leq \left[ \frac{1}{4} + \frac{(x - \frac{a+b}{2})^2}{(b-a)^2} \right] (b-a), \]
and hence the Ostrowski’s inequality can be extended in the following way:

Let \( f \) be a differentiable mapping with bounded \( f' \) on the interior \( I \) of an open interval \( I \subset \mathbb{R} \). Then for all \( a, b \in I \) with \( b > a \) and for all \( x \in I \), holds
\[ |f(x) - \frac{1}{b-a} \int_a^b f(t) \, dt| \leq \left[ \frac{1}{4} + \frac{(x - \frac{a+b}{2})^2}{(b-a)^2} \right] (b-a) \sup_{x \in I} |f'(x)|. \quad (2.19) \]
Since (2.18) holds, we conclude that the inequality (2.17) is better than the restriction of inequality (2.19) on \( I - (a,b) \).

### 3. Generalized midpoint inequalities

Now, we will prove two generalized midpoint integral inequalities:

**Theorem 6.** Let \( f \in C^n([a,b]) \), and \( f^{(n+1)} \) exists and is bounded on \( (a,b) \). Then, if \( n \) is odd we have
\[ \gamma_{n+1} \frac{(b-a)^{n+2}}{2^{n+1}(n+2)!} \leq \int_a^b f(t) \, dt - \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \frac{f^{(i)}}{2^i} \left( \frac{a+b}{2} \right)^{i+1} \frac{(b-a)^{n+1}}{2^{i+1}(i+1)!} \]
\[ \leq \Gamma_{n+1} \frac{(b-a)^{n+2}}{2^{n+1}(n+2)!}. \quad (3.1) \]
where \( \gamma_{n+1} := \inf_{x \in (a,b)} f^{(n+1)}(x) \), \( \Gamma_{n+1} := \sup_{x \in (a,b)} f^{(n+1)}(x) \). The inequality (3.1) is sharp in the sense that the constant \( \frac{1}{2^{n+1}(n+2)!} \) cannot be replaced by a smaller one.

**Proof.** From (2.15) by \( x = \frac{a+b}{2} \), we get easily (3.1). Similarly as in Theorem 4, we can prove that the equality in (3.1) holds by \( f(x) := x^{n+1} \), and hence the sharpness of inequality is verified. \( \square \)

**Remark 3.** For \( n = 1 \), from (3.1) we get (1.3).

**Theorem 7.** Let \( f \) be as in Theorem 6. If \( n \) is even, then holds

\[
\left| \int_{a}^{b} f(t) \, dt - \sum_{i=0}^{n} f^{(i)} \left( \frac{a + b}{2} \right) \frac{(b-a)^{i+1}}{2!(i+1)!} \right| \leq \frac{(\Gamma_{n+1} - \Gamma_{n+1}^-)(b-a)^{n+2}}{2^{n+2}(n+2)!}. \tag{3.2}
\]

**Proof.** Let \( t \) be any number in \((a, b)\). Applying the Taylor-Lagrange formula to \( f \) about \( \frac{a+b}{2} \) we

\[
R_n \left( f; \frac{a+b}{2}, t \right) = \frac{(t-\frac{a+b}{2})^{n+1}}{(n+1)!} f^{(n+1)}(\xi),
\]

for some \( \xi \in \left( \left\{ t, \frac{a+b}{2} \right\} \right) \).

That is

\[
\frac{(t-\frac{a+b}{2})^{n+1}}{(n+1)!} \inf_{x \in \left( t, \frac{a+b}{2} \right)} f^{(n+1)}(x) \leq R_n \left( f; \frac{a+b}{2}, t \right) \leq \frac{(t-\frac{a+b}{2})^{n+1}}{(n+1)!} \sup_{x \in \left( t, \frac{a+b}{2} \right)} f^{(n+1)}(x). \tag{3.3}
\]

Moreover, we have

\[
\gamma_{n+1} \leq \inf_{x \in \left( t, \frac{a+b}{2} \right)} f^{(n+1)}(x) \leq \sup_{x \in \left( t, \frac{a+b}{2} \right)} f^{(n+1)}(x) \leq \Gamma_{n+1}. \tag{3.4}
\]

If \( t > \frac{a+b}{2} \), then \( (t - \frac{a+b}{2})^{n+1} > 0 \), and hence from (3.3), (3.4), we get

\[
\gamma_{n+1} \frac{(t-\frac{a+b}{2})^{n+1}}{(n+1)!} \leq R_n \left( f; \frac{a+b}{2}, t \right) \leq \Gamma_{n+1} \frac{(t-\frac{a+b}{2})^{n+1}}{(n+1)!}. \tag{3.5}
\]
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Now, integrating (3.5) over \([\frac{a+b}{2}, b]\), we obtain

\[
\gamma_{n+1} \frac{(t - \frac{a+b}{2})^{n+1}}{(n+2)!2^{n+2}} \leq \int_{\frac{a+b}{2}}^{b} f(t) dt - \sum_{i=0}^{n} \frac{(b-a)^{i+1} f^{(i)} \left( \frac{a+b}{2} \right)}{(i+1)!2^{i+1}}
\]

\[
\leq \Gamma_{n+1} \frac{(b-a)^{n+2}}{(n+1)!2^{n+2}}.
\]

(3.6)

If \(t < \frac{a+b}{2}\), then since \(n\) is even we have \((t - \frac{a+b}{2})^{n+1} < 0\), and hence from (3.3), (3.4) we result in:

\[
\Gamma_{n+1} \frac{(t - \frac{a+b}{2})^{n+1}}{(n+1)!} \leq R_{n} \left( f; \frac{a+b}{2}, t \right) \leq \gamma_{n+1} \frac{(t - \frac{a+b}{2})^{n+1}}{(n+1)!}.
\]

(3.7)

Now, integrating (3.7) over \([a, \frac{a+b}{2}]\), we obtain

\[
-\gamma_{n+1} \frac{(b-a)^{n+2}}{(n+2)!2^{n+2}} \leq \int_{a}^{\frac{a+b}{2}} f(t) dt - \sum_{i=0}^{n} \frac{(-1)^i (b-a)^{i+1} f^{(i)} \left( \frac{a+b}{2} \right)}{(n+1)!2^{i+1}}
\]

\[
\leq -\gamma_{n+1} \frac{(b-a)^{n+2}}{(n+1)!2^{n+2}}.
\]

(3.8)

Adding (3.6) and (3.8), we obtain

\[
-\frac{(\gamma_{n+1} - \gamma_{n+1})(b-a)^{n+2}}{(n+2)!2^{n+2}} \leq \int_{a}^{b} f(t) dt - \sum_{i=0}^{n} \frac{(b-a)^{i+1} f^{(i)} \left( \frac{a+b}{2} \right)}{(i+1)!2^{i}}
\]

\[
\leq \frac{(\Gamma_{n+1} - \gamma_{n+1})(b-a)^{n+2}}{(n+2)!2^{n+2}},
\]

that is (3.2).

\[\square\]

Remark 4. Applying the estimation (3.2) by \(n = 0\) we obtain:

Let \(f \in C([a, b])\) be a mapping such that \(f'\) is bounded on \((a, b)\).

Then holds

\[
\left| f \left( \frac{a+b}{2} \right) - \frac{1}{b-a} \int_{a}^{b} f(t) dt \right| \leq \frac{(\Gamma_{1} - \gamma_{1})(b-a)}{8},
\]

where \(\gamma_{1} := \inf_{x \in (a, b)} f'(x), \Gamma_{1} := \sup_{x \in (a, b)} f'(x)\). Obviously, if \(f\) is convex then the inequalities (1.4), (3.9) are identical. The advantage by inequality (3.9) is that the assumptions, \(f\) is twice differentiable on \((a, b)\), and \(f'' \in L_{1}(a, b)\) are not necessary.
References


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