On some sum form functional equations related to information theory

Prem Nath

Department of Mathematics
University of Delhi
Delhi 110 007
India

Abstract
This paper deals with some sum form functional equations related to the Shannon entropy and the nonadditive entropies of degree $\alpha$.

Keywords and phrases: Functional equations, Lebesgue measurable solutions, the Shannon entropy, the nonadditive entropy of degree $\alpha$.

1. Introduction

For $n = 1, 2, 3, \ldots$, let

$$\Gamma_n = \left\{ (p_1, \ldots, p_n) : p_i \geq 0, i = 1, \ldots, n ; \sum_{i=1}^{n} p_i = 1 \right\}$$

denote the set of all $n$-component complete discrete probability distributions with nonnegative elements.

For $(p_1, \ldots, p_n) \in \Gamma_n$, $n = 1, 2, 3, \ldots$; the entropies

$$H_n(p_1, \ldots, p_n) = -\sum_{i=1}^{n} p_i \log_2 p_i$$

are known as the Shannon entropies [15] with $H_n : \Gamma_n \rightarrow \mathbb{R}$; $\mathbb{R}$ denoting the set of all real numbers; $n = 1, 2, 3, \ldots$, and $0 \log_2 0 = 0$. These
entropies are additive. Also, for \((p_1, \ldots, p_n) \in \Gamma_n, n = 1, 2, 3, \ldots\); the entropies
\[
H_{\alpha}^n(p_1, \ldots, p_n) = (1 - 2^{1-\alpha})^{-1} \left(1 - \sum_{i=1}^{n} p_i^\alpha\right)
\]
(1.2)
with \(H_{\alpha}^n : \Gamma_n \rightarrow \mathbb{R}, n = 1, 2, 3, \ldots\); and \(0^\alpha := 0\) for all \(\alpha > 0\), are called the entropies of degree \(\alpha, \alpha > 0, \alpha \neq 1\). These entropies are nonadditive and were given by J. Havrda and F. Charvat [5].

Both the measures of entropies (1.1) and (1.2) admit of sum representations.

The functional equation
\[
\sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i y_j) = \sum_{i=1}^{m} f(x_i) + \sum_{j=1}^{n} f(y_j) + \lambda \sum_{i=1}^{m} f(x_i) \sum_{j=1}^{n} f(y_j)
\]
(1.3)
with \((x_1, \ldots, x_m) \in \Gamma_m, (y_1, \ldots, y_n) \in \Gamma_n, \lambda \in \mathbb{R}\) is closely related to (1.1) and (1.2). When \(\lambda = 0\), it reduces to the functional equation
\[
\sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i y_j) = \sum_{i=1}^{m} f(x_i) + \sum_{j=1}^{n} f(y_j)
\]
(1.4)
considered first by T. W. Chaundy and J. B. Mcleod [3].

During the last four decades, enough research work, concerning (1.3) and (1.4), has been done by several researchers (see [4], [6], [8] to [12], [13] etc.). The solutions of (1.3) and (1.4) have been obtained by imposing various regularity conditions such as continuity, measurability etc. on \(f\) and, in some cases, without imposing any restriction on \(f\) but by imposing the restrictions on the positive integers \(m\) and \(n\).

PL. Kannappan [6] obtained the solutions of (1.3) and (1.4) by assuming \(f\) to be measurable, in the sense of Lebesgue, on the closed interval \(I = [0, 1]\) and taking \(m \geq 3, n \geq 3\) to be fixed integers. He [7] also considered the functional equation
\[
\sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i y_j) = \sum_{i=1}^{m} g(x_i) + \sum_{j=1}^{n} h(y_j) + \sum_{i=1}^{m} k(x_i) \sum_{j=1}^{n} f(y_j)
\]
(1.5)
with \((x_1, \ldots, x_m) \in \Gamma_m, (y_1, \ldots, y_n) \in \Gamma_n\). Recently, the author [14] has obtained the measurable solutions of (1.5) by assuming \(f : I \rightarrow \mathbb{R}\),
g : I \to \mathbb{R}, h : I \to \mathbb{R}, k : I \to \mathbb{R}, \ell : I \to \mathbb{R}, I = [0,1], to be measurable on I in the sense of Lebesgue and \((x_1, \ldots, x_m) \in \Gamma_m, (y_1, \ldots, y_n) \in \Gamma_n, m,n = 1,2,3,\ldots,\) without making use of Satz 2 of E. Vincze [16] and also mentioned that the Lebesgue measurable solutions of (1.5) with \((x_1, \ldots, x_m) \in \Gamma_m, (y_1, \ldots, y_n) \in \Gamma_n, m \geq 3, n \geq 3\) fixed integers, will be presented in the subsequent research work.

This task has been accomplished by proving Theorem 1 in section 2 of this paper without making use of Satz 2 of E. Vincze [16]. The obtained solutions depend upon the fixed integers \(m \geq 3, n \geq 3\) and also contain several real suitably restricted constants. Their forms are quite complicated and it seems undesirable to deduce the Lebesgue measurable solutions of (1.3) from these solutions. In Theorem 2, proved in section 3 of this paper, the Lebesgue measurable solutions of (1.3), for \(\lambda \neq 0\) and all \((x_1, \ldots, x_m) \in \Gamma_m, (y_1, \ldots, y_n) \in \Gamma_n, m \geq 3, n \geq 3\) fixed integers, have also been obtained.

We may mention that PL. Kannappan [7] obtained the measurable solutions of (1.5) by taking \(m \geq 3, n \geq 3\) to be fixed integers and assuming \(f, g, h, k\) and \(\ell\) to be real-valued Lebesgue measurable on \(I\) and summarized them in the only theorem mentioned on p. 49 in [7] and making use of Satz 2 of E. Vincze [16]. Theorem 1, proved in this paper, is absolutely different from the one mentioned on p. 49 in [7], the reason being that it has been proved by not making use of Satz 2 of E. Vincze [16]. Besides, this theorem also throws some light on the measurable but discontinuous solutions.

The functional equation (1.3) is a useful particular case of (1.5). Theorem 2, proved in this paper, should be considered as an improved version of Theorem 2 mentioned on p. 149 in [6].

Throughout this paper, by a measurable function, we shall mean a real-valued Lebesgue measurable function.

2. The measurable solutions of (1.5)

In this section, we prove the following:

**Theorem 1.** If \(f : I \to \mathbb{R}, g : I \to \mathbb{R}, h : I \to \mathbb{R}, k : I \to \mathbb{R}\) and \(\ell : I \to \mathbb{R}, I = [0,1]\) are measurable on \(I\) in the sense of Lebesgue and satisfy the functional equation (1.5) for all \((x_1, \ldots, x_m) \in \Gamma_m, (y_1, \ldots, y_n) \in \Gamma_n, m \geq 3, n \geq 3\)
fixed integers; then (1.5) has the following solutions:

\[
\begin{align*}
(f(x) &= \lambda x \log x + (\alpha_1 - \beta_1 mn + \beta_2 m + \beta_3 n + \beta_4 \beta_5 mn)x + \beta_1 \\
g(x) &= \lambda x \log x + (\alpha_2 - \alpha_4 \alpha_5 n)x + \beta_2 \\
h(x) &= \lambda x \log x - (\alpha_4 + \beta_4 m)\xi(x) + (\alpha_3 - \beta_4 \alpha_5 m)x + \beta_3 \\
k(x) &= \alpha_4 x + \beta_4 \\
\ell(x) &= \xi(x) + \alpha_5 x + \beta_5 \\
\end{align*}
\]

\[
\begin{align*}
(\xi(x) &= \lambda x \log x + (\alpha_1 - \beta_1 mn + \beta_2 m + \beta_3 n + \beta_4 \beta_5 mn)x + \beta_1 \\
g(x) &= \lambda x \log x - (\alpha_5 - \beta_5 n)\xi(x) + (\alpha_2 - \beta_5 \alpha_4 n)x + \beta_2 \\
h(x) &= \lambda x \log x + (\alpha_3 - \beta_4 \alpha_5 m)x + \beta_3 \\
k(x) &= \xi(x) + \alpha_4 x + \beta_4 \\
\ell(x) &= \alpha_5 x + \beta_5 \\
\end{align*}
\]

\[
\begin{align*}
(f(x) &= \frac{1}{2}c\lambda_1^2 x(\log x)^2 + \lambda x \log x \\
&+ (\alpha_1 - \beta_1 mn + \beta_2 m + \beta_3 n + \beta_4 \beta_5 mn)x + \beta_1 \\
g(x) &= \frac{1}{2}c\lambda_1^2 x(\log x)^2 + (\lambda - \alpha_5 \lambda_1 - \beta_5 n \lambda_1)x \log x \\
&+ (\alpha_2 - \beta_5 \alpha_4 n)x + \beta_2 \\
h(x) &= \frac{1}{2}c\lambda_1^2 x(\log x)^2 + (\lambda - \alpha_4 c\lambda_1 - \beta_4 mc \lambda_1)x \log x \\
&+ (\alpha_3 - \beta_4 \alpha_5 m)x + \beta_3 \\
k(x) &= \lambda x \log x + \alpha_4 x + \beta_4 \\
\ell(x) &= c\lambda_1 x \log x + \alpha_5 x + \beta_5 \\
\end{align*}
\]

\[
\begin{align*}
(f(x) &= \lambda x \log x + (\alpha_1 - \mu^2 - \beta_1 mn + \beta_2 m + \beta_3 n + \beta_4 \beta_5 mn)x + \beta_1 \\
if 0 \leq x < 1 \\
&= \alpha_1 - (\beta_1 mn - \beta_2 m - \beta_3 n - \beta_4 \beta_5 mn) + \beta_1 \text{ if } x = 1 \\
g(x) &= \lambda x \log x + (\alpha_5 \mu - \mu^2 + \alpha_2 - \beta_5 \alpha_4 n + \beta_4 \mu n)x + \beta_2 \\
if 0 \leq x < 1 \\
&= \alpha_2 - \beta_5 \alpha_4 n + \beta_2 \text{ if } x = 1 \\
h(x) &= \lambda x \log x + (\alpha_4 \mu + \alpha_5 - \mu^2 - \beta_4 \alpha_5 m + \beta_4 mc \mu)x + \beta_3 \\
if 0 \leq x < 1 \\
&= \alpha_3 - \beta_4 \alpha_5 m + \beta_3 \text{ if } x = 1 \\
k(x) &= (\alpha_4 - \mu)x + \beta_4 \text{ if } 0 \leq x < 1 \\
&= \alpha_4 + \beta_4 \text{ if } x = 1 \\
\ell(x) &= (\alpha_5 - \mu)x + \beta_5 \text{ if } 0 \leq x < 1 \\
&= \alpha_5 + \beta_5 \text{ if } x = 1 \\
\end{align*}
\]
for all \( x \in \mathbb{R} \)
\[ f(x) = \lambda x \log x + c \mu^2 x^{\delta + 1} \]
\[ + (\alpha_1 - c \mu^2 - \beta_1 mn + \beta_2 m + \beta_3 n + \beta_4 \beta_5 mn) x + \beta_1 \]
if \( 0 < x \leq 1 \)
\[ = \beta_1 \] if \( x = 0 \)

\[ g(x) = \lambda x \log x + (c \mu^2 - \alpha_5 \mu - \beta_5 n \mu) x^{\delta + 1} \]
\[ + (\alpha_2 + \alpha_5 \mu - c \mu^2 - \beta_3 \alpha_4 n + \beta_3 n \mu) x + \beta_2 \]
if \( 0 < x \leq 1 \)
\[ = \beta_2 \] if \( x = 0 \)

\[ h(x) = \lambda x \log x + (c \mu^2 - \alpha_4 \mu - \beta_4 m \mu) x^{\delta + 1} \]
\[ + (\alpha_3 - c \mu^2 + \alpha_4 \mu - \beta_4 \alpha_5 m + \beta_4 m \mu) x + \beta_3 \]
if \( 0 < x \leq 1 \)
\[ = \beta_3 \] if \( x = 0 \)

\[ k(x) = \mu x^{\delta + 1} + (\alpha_4 - \mu) x + \beta_4 \]
if \( 0 < x \leq 1 \)
\[ = \beta_4 \] if \( x = 0 \)

\[ \ell(x) = c \mu x^{\delta + 1} + (\alpha_5 - c \mu) x + \beta_5 \]
if \( 0 < x \leq 1 \)
\[ = \beta_5 \] if \( x = 0 \)

where

\[ \beta_1 = f(0), \beta_2 = g(0), \beta_3 = h(0), \beta_4 = k(0), \beta_5 = \ell(0) \] (2.1)

\[ \alpha_1 = \alpha_2 + \alpha_3 + \alpha_4 \alpha_5 \] (2.2)

\[ \xi : I \to \mathbb{R}, \zeta : I \to \mathbb{R} \] are arbitrary measurable functions

with \( \xi(1) = 0, \zeta(1) = 0 \) (2.3)

\( \lambda \) an arbitrary real constant and \( \lambda_1, c, \mu, \delta \) arbitrary nonzero real constants and

\[ 0 \log 0 = 0(\log 0)^2 = 0, \ 0^\beta = 0 \] only for \( \beta > 0 \). (2.4)

The proof of this theorem needs the following:

**Lemma 1 (PL. Kannappan [6]).** Let \( \varphi : I \times I \to \mathbb{R}, I = [0, 1] \) be measurable in each variable in the sense of Lebesgue and satisfy the functional equation

\[ \sum_{i=1}^{m} \sum_{j=1}^{n} \varphi(x_i, y_j) = 0 \] (2.5)

for all \( (x_1, \ldots, x_m) \in \Gamma_m, (y_1, \ldots, y_n) \in \Gamma_n, m \geq 3, n \geq 3 \) being fixed integers. Then \( \varphi \) is of the form

\[ \varphi(x, y) = \varphi(0, y)(1-mx) + \varphi(x, 0)(1-ny) - \varphi(0, 0)(1-mx)(1-ny) \] (2.6)

for all \( x \in I, y \in I \).
Hence, for all $x \in I$, $y \in I$, (2.6) reduces to
\[
\begin{align*}
\phi(x, y) &= f(xy) - yg(x) - xh(y) - k(x)\ell(y) \\
&= [\beta_1 - \beta_2y - \beta_4\ell(y)][1 - mx] + [\beta_1 - \beta_3x - \beta_5k(x)][1 - ny] \\
&\quad - (\beta_1 - \beta_4\beta_5)(1 - mx)(1 - ny)
\end{align*}
\]
which can be written in the form
\[
f(xy) + (\beta_1 mn - \beta_2 m - \beta_3 n - \beta_4 \beta_5 mn)xy - \beta_1
= \left[ g(x) - \beta_4 \beta_5 n + \beta_5 nk(x) - \beta_2 \right] y + \left[ h(y) - \beta_4 \beta_5 m + \beta_4 m \ell(y) - \beta_3 \right] x
+ \left[ k(x) - \beta_4 \right] \left[ \ell(y) - \beta_5 \right].
\] (2.11)

Let us define \( F : I \to \mathbb{R}, G : I \to \mathbb{R}, H : I \to \mathbb{R}, K : I \to \mathbb{R} \) and \( L : I \to \mathbb{R} \) as
\[
\begin{align*}
F(x) &= f(x) + (\beta_1 mn - \beta_2 m - \beta_3 n - \beta_4 \beta_5 mn)x - \beta_1, \\
G(x) &= g(x) - \beta_4 \beta_5 n + \beta_5 nk(x) - \beta_2, \\
H(x) &= h(x) - \beta_4 \beta_5 m + \beta_4 m \ell(x) - \beta_3, \\
K(x) &= k(x) - \beta_4, \\
L(x) &= \ell(x) - \beta_5.
\end{align*}
\] (2.12)

Then (2.12) reduces to the functional equation
\[
F(xy) = yG(x) + xH(y) + K(x)L(y)
\] (2.13)
with \( x \in I, y \in I \). From (2.9) and (2.12), it follows that
\[
F(1) = \alpha_1, \; G(1) = \alpha_2, \; H(1) = \alpha_3, \; K(1) = \alpha_4, \; L(1) = \alpha_5.
\] (2.14)

Moreover, from (2.1) and (2.12) it follows that
\[
F(0) = G(0) = H(0) = K(0) = L(0) = 0.
\] (2.15)

From (2.12), one can easily conclude that
\[
\begin{align*}
f(x) &= F(x) - (\beta_1 mn - \beta_2 m - \beta_3 n - \beta_4 \beta_5 mn)x + \beta_1, \\
g(x) &= G(x) - \beta_5 n K(x) + \beta_2, \\
h(x) &= H(x) - \beta_4 m L(x) + \beta_3, \\
k(x) &= K(x) + \beta_4, \\
\ell(x) &= L(x) + \beta_5.
\end{align*}
\] (2.16)
where \( F, G, H, K \) and \( L \) satisfy the functional equation (2.13) with \( x \in I, y \in I \).

The measurability of \( f, g, h, k \) and \( \ell \), on \( I \), imply the measurability of \( F, G, H, K \) and \( L \) defined in (2.12). The measurable solutions of (2.13), subject to the conditions (2.14), (2.2) and (2.3), have recently been obtained
in [14]. These solutions are:

\[
\begin{align*}
F(x) &= \lambda x \log x + (\alpha_1 - cd^2)x + cd^2 \\
G(x) &= \lambda x \log x + (\alpha_2 + \alpha_3 d - cd^2)x + (cd^2 - \alpha_5 d) \\
H(x) &= \lambda x \log x + (\alpha_3 + \alpha_4 cd - cd^2)x + (cd^2 - \alpha_4 cd) \\
K(x) &= (\alpha_4 - d)x + d \\
L(x) &= (\alpha_5 - cd)x + cd \\
F(x) &= \lambda x \log x + \alpha_1 x \\
G(x) &= \lambda x \log x + \alpha_2 x \\
H(x) &= \lambda x \log x - \alpha_5 \xi(x) + \alpha_3 x \\
K(x) &= \alpha_4 x \\
L(x) &= \xi(x) + \alpha_5 x, \quad \xi \text{ as in (2.3)}. \\
F(x) &= \lambda x \log x + \alpha_1 x \\
G(x) &= \lambda x \log x - \alpha_5 \xi(x) + \alpha_2 x \\
H(x) &= \lambda x \log x + \alpha_3 x \\
K(x) &= \xi(x) + \alpha_4 x, \quad \xi \text{ as in (2.3)} \\
L(x) &= \alpha_5 x \\
F(x) &= \frac{1}{2} c\lambda_1^2 x (\log x)^2 + \lambda x \log x + \alpha_1 x \\
G(x) &= \frac{1}{2} c\lambda_1^2 x (\log x)^2 + (\lambda - \alpha_3 \lambda_1) x \log x + \alpha_2 x \\
H(x) &= \frac{1}{2} c\lambda_1^2 x (\log x)^2 + (\lambda - \alpha_4 \lambda_1) x \log x + \alpha_3 x \\
K(x) &= \lambda_1 x \log x + \alpha_4 x \\
L(x) &= c\lambda_1 x \log x + \alpha_5 x \\
F(x) &= \lambda x \log x + (\alpha_1 - c\mu^2)x \quad \text{if } 0 \leq x < 1 \\
&= \alpha_1 \quad \text{if } x = 1 \\
G(x) &= \lambda x \log x + (\alpha_3 \mu - c\mu^2 + \alpha_2)x \quad \text{if } 0 \leq x < 1 \\
&= \alpha_2 \quad \text{if } x = 1 \\
H(x) &= \lambda x \log x + (\alpha_4 \mu + \alpha_3 - c\mu^2)x \quad \text{if } 0 \leq x < 1 \\
&= \alpha_3 \quad \text{if } x = 1 \\
K(x) &= (\alpha_4 - \mu)x \quad \text{if } 0 \leq x < 1 \\
&= \alpha_4 \quad \text{if } x = 1 \\
L(x) &= (\alpha_5 - c\mu)x \quad \text{if } 0 \leq x < 1 \\
&= \alpha_5 \quad \text{if } x = 1
\end{align*}
\]
and

\[
\begin{align*}
F(x) &= \lambda x \log x + c \mu^2 x^{\beta+1} + (\alpha_1 - c \mu^2)x \quad \text{if } 0 < x \leq 1 \\
&= 0 \quad \text{if } x = 0 \\
G(x) &= \lambda x \log x + (c \mu^2 - \alpha_3 \mu)x^{\beta+1} \\
&\quad + (\alpha_2 + \alpha_5 \mu - c \mu^2)x \quad \text{if } 0 < x \leq 1 \\
&= 0 \quad \text{if } x = 0 \\
H(x) &= \lambda x \log x + (c \mu^2 - \alpha_4 \mu)x^{\beta+1} \\
&\quad + (\alpha_3 - c \mu^2 + \alpha_4 \mu)x \quad \text{if } 0 < x \leq 1 \\
&= 0 \quad \text{if } x = 0 \\
K(x) &= \mu x^{\beta+1} + (\alpha_4 - \mu)x \quad \text{if } 0 < x \leq 1 \\
&= 0 \quad \text{if } x = 0 \\
L(x) &= c \mu x^{\beta+1} + (\alpha_5 - c \mu)x \quad \text{if } 0 < x \leq 1 \\
&= 0 \quad \text{if } x = 0.
\end{align*}
\]

(2.22)

In (2.17) to (2.22), \( \lambda \) is an arbitrary real constant whereas \( \lambda_1, c, \mu, \delta, d \) are arbitrary nonzero real constants; \( \alpha_1 = \alpha_2 + \alpha_3 + \alpha_4 \alpha_5 \) and \( 0 \log 0 = 0(\log 0)^2 = 0; 0^\beta = 0 \) only for \( \beta > 0 \).

We reject (2.17) as, in this case, (2.15) is not satisfied. From (2.16), and (2.18) to (2.22), the required solutions \((s)_1\) to \((s)_5\) of (1.5), can be derived. The tedious calculations are omitted. It can be verified that \((s)_1\) to \((s)_5\) do satisfy the functional equation (1.5) subject to the conditions (2.1) to (2.4). This completes the proof of Theorem 1. Notice that, to prove Theorem 1, we have not made use of Satz 2 proved by E. Vincze [16].

It would be worth while to remark that, in \((s)_4\), all the measurable functions \( f, g, h, k \) and \( \ell \) are discontinuous at \( x = 1 \). On the other hand, in \((s)_5\), all the measurable functions \( f, g, h, k \) and \( \ell \) are discontinuous at \( x = 0 \) provided \( \delta + 1 \leq 0 \).

To proceed further, we need the following:

Lemma 2 (L. Losonczi [8]). If a mapping \( g : I \to \mathbb{R}, I = [0, 1] \), is measurable on \( I \) and satisfies the equation

\[
\sum_{i=1}^{m} g(x_i) = 0
\]

(2.23)

for all \((x_1, \ldots, x_m) \in \Gamma_m, m \geq 3 \) a fixed integer; then \( g \) is the form

\[
g(x) = g(0)(1 - mx), \quad x \in I
\]

(2.24)

where \( g(0) \) is an arbitrary real constant.
3. The measurable solutions of (1.3)

In this section, our task is to obtain the Lebesgue measurable solutions of the functional equation (1.3) for \( \lambda \neq 0 \) and all \((x_1, \ldots, x_m) \in \Gamma_m, (y_1, \ldots, y_n) \in \Gamma_n \) where \( m \geq 3 \) and \( n \geq 3 \) are fixed integers.

Due to the complicated forms of the solutions \((s_1)\) to \((s_5)\) of (1.5), it does not seem desirable to deduce the solutions of (1.3) from these solutions subject to the conditions (2.1) to (2.4). Below, we give a simple method of finding the measurable solutions of (1.3) for \( \lambda \neq 0 \) and all \((x_1, \ldots, x_m) \in \Gamma_m, (y_1, \ldots, y_n) \in \Gamma_n \) where \( m \geq 3 \) and \( n \geq 3 \) are fixed integers. We prove the following:

**Theorem 2.** If \( f : I \to \mathbb{R}, I = [0, 1], \) is measurable on 1 in the sense of Lebesgue and satisfies the functional equation (1.3) for \( \lambda \neq 0 \) and all \((x_1, \ldots, x_m) \in \Gamma_m, (y_1, \ldots, y_n) \in \Gamma_n \) with \( m \geq 3 \) and \( n \geq 3 \) fixed integers, then (1.3) has the following solutions:

\[
f(x) = 0, \quad 0 \leq x \leq 1
\]  
(3.1)

\[
f(x) = \begin{cases} 
-x \lambda & \text{if } 0 \leq x < 1 \\
0 & \text{if } x = 1
\end{cases}
\]  
(3.2)

\[
f(x) = \begin{cases} 
x^\delta - x & \text{if } 0 < x \leq 1 \\
0 & \text{if } x = 0
\end{cases}
\]  
(3.3)

and

\[
f(x) = f(0) + (f(1) - f(0))x, \quad 0 \leq x \leq 1
\]  
(3.4)

where \( \delta \neq 0 \), \( f(0) \) and \( f(1) \) are arbitrary real constants.

**Proof.** Let us put \( y_1 = 1, y_2 = y_3 = \ldots = y_n = 0 \) in (1.3). We obtain

\[
(mn - m)f(0) = \left( \lambda \sum_{i=1}^{m} f(x_i) + 1 \right) (f(1) + (n - 1)f(0)).
\]  
(3.5)

Case 1. \( f(1) + (n - 1)f(0) = 0 \). Then (3.5) gives \( f(0) = 0 \) as \( m \geq 3 \) and \( n \geq 3 \). Hence, \( f(1) = 0 \). Thus, we have \( f(0) = f(1) = 0 \).

Let us define a mapping \( \varphi : I \times I \to \mathbb{R} \) as

\[
\varphi(x, y) = f(xy) - yf(x) - xf(y) - \lambda f(x)f(y)
\]
for all $x \in I, y \in I$. Then, (1.3) reduces to (2.5). By Lemma 1, (2.6) holds with
\[
\varphi(0,y) = f(0) - yf(0) - \lambda f(0)f(y) \\
\varphi(x,0) = f(0) - xf(0) - \lambda f(x)f(0) \\
\varphi(0,0) = f(0) - \lambda[f(0)]^2.
\]
Hence
\[
f(xy) - yf(x) - xf(y) - \lambda f(x)f(y) \\
= (f(0) - yf(0) - \lambda f(0)f(y))(1 - mx) + (f(0) - xf(0) - \lambda f(x)f(0)) \\
\times (1 - ny) + \{\lambda[f(0)]^2 - f(0)\}(1 - mx)(1 - ny). 
\tag{3.6}
\]
Substituting $f(0) = 0$ in (3.6) we obtain
\[
f(xy) = yf(x) + xf(y) + \lambda f(x)f(y) 
\tag{3.7}
\]
for all $x \in I, y \in I$. Since the values of $f$ at 0 and 1 are known, we consider the functional equation
\[
f(xy) = yf(x) + xf(y) + \lambda f(x)f(y), \quad x \in ]0,1[, \ y \in ]0,1[. 
\tag{3.8}
\]
Define a mapping $h : ]0,1[ \rightarrow \mathbb{R}$ as
\[
h(x) = 1 + \frac{\lambda f(x)}{x} 
\tag{3.9}
\]
for all $x \in ]0,1[$. Then (3.8) reduces to the equation
\[
h(xy) = h(x)h(y), \quad x \in ]0,1[, \ y \in ]0,1[. 
\tag{3.10}
\]
The measurability of $f$ on $I$ implies the measurability of $h$ on $]0,1[$. So, $h$ is of the forms
\[
h(x) = 0; \quad h(x) = 1; \quad h(x) = \delta, \quad \delta \neq 0
\]
where $\delta \neq 0$ is an arbitrary real constant.
If $h(x) = 0, x \in ]0,1[$, then (3.9) gives $f(x) = -\frac{x}{\lambda}$ for all $x \in ]0,1[$. Since $f(0) = f(1) = 0$, we get the measurable solution (3.2). This measurable solution is discontinuous at $x = 1$.
If $h(x) = 1, x \in ]0,1[$, then (3.9) gives $f(x) = 0$ for all $x \in ]0,1[$. Since $f(0) = f(1) = 0$, we get the measurable solution (3.1).
If \( h(x) = x^\delta, \delta \neq 0, x \in ]0,1[; \) then (3.9) gives \( f(x) = \frac{x^{\delta+1} - x}{\lambda} \) for all \( x \in ]0,1[, \delta \neq 0 \) being an arbitrary real constant. Since \( f(0) = f(1) = 0 \), we get the measurable solution (3.3). This solution is discontinuous at \( x = 0 \) for \( \delta + 1 \leq 0 \) and continuous at \( x = 0 \) for \( \delta + 1 > 0 \).

**Case 2.** \( f(1) + (n-1)f(0) \neq 0 \). In this case, (3.5) reduces to

\[
\lambda \sum_{i=1}^{m} f(x_i) + 1 = \frac{m(n-1)f(0) - f(1)}{f(1) + (n-1)f(0)}
\]

which can be written in the form

\[
\sum_{i=1}^{m} \left\{ \lambda f(x_i) - \frac{(m-1)(n-1)f(0) - f(1)}{f(1) + (n-1)f(0)} x_i \right\} = 0.
\]

Making use of Lemma 2, it follows that

\[
\lambda f(x) - \frac{(m-1)(n-1)f(0) - f(1)}{f(1) + (n-1)f(0)} x = \lambda f(0)(1 - mx)
\]

(3.11)

for all \( x \in I \).

Now let us choose \( x_1 = 1, x_2 = x_3 = \ldots = x_m = 0 \) and \( y_1 = 1, y_2 = y_3 = \ldots = y_n = 0 \) in (1.3), we obtain

\[
f(1) + (mn-1)f(0)
\]

\[
= \{f(1) + (m-1)f(0)\} + \{f(1) + (n-1)f(0)\}
\]

\[
+ \lambda \{f(1) + (m-1)f(0)\} \{f(1) + (n-1)f(0)\}
\]

(3.12)

from which it follows that

\[
\frac{1}{\lambda} \left\{ \frac{(m-1)(n-1)f(0) - f(1)}{f(1) + (n-1)f(0)} \right\} = f(1) + (m-1)f(0).
\]

(3.13)

From (3.11) and (3.13), (3.4) follows. This completes the proof of Theorem 2. \( \square \)

**Some comments on solution (3.4).** It is easy to verify that (3.1) to (3.4) do satisfy (1.3). As regards (3.4) is concerned, it satisfies (1.3) subject to (3.12) which holds, too, and hence there is no need to assume it. Below, we discuss some important special cases of (3.4).

If \( f(0) = 0 \) but \( f(1) \neq 0 \), then (3.4) reduces to

\[
f(x) = f(1)x
\]

(3.14)
for all \( x, \ 0 \leq x \leq 1 \). On the other hand, if we put \( f(0) = 0 \) in (3.12) and use the fact that \( f(1) \neq 0 \), then we obtain \( f(1) = -\frac{1}{\lambda} \) and (3.14) reduces to
\[
f(x) = -\frac{x}{\lambda}, \quad 0 \leq x \leq 1. \tag{3.15}
\]
It can be easily verified that (3.15) satisfies (1.3).

If \( f(1) = 0 \) but \( f(0) \neq 0 \), then (3.4) reduces to
\[
f(x) = f(0)(1 - x), \quad 0 \leq x \leq 1. \tag{3.16}
\]
If we put \( f(1) = 0 \) in (3.12) and make use of the facts that \( f(0) \neq 0 \) and \( m \geq 3, \ n \geq 3 \) fixed integers, then it follows that \( f(0) = \frac{1}{\lambda} \) and (3.16) reduces to
\[
f(x) = \frac{1 - x}{\lambda}, \quad 0 \leq x \leq 1. \tag{3.17}
\]
It can be verified that (3.17) also satisfies (1.3).

If \( f(0) \neq 0, \ f(1) \neq 0, \ f(1) + (n - 1)f(0) \neq 0 \), then it seems better to write (3.4) in the form
\[
f(x) = xf(1) + (1 - x)f(0), \quad 0 \leq x \leq 1
\]
so that \( f(x) \) can be regarded as a convex linear combination of \( f(1) \) and \( f(0) \).

Below, we give an alternative method of deriving (3.4).

Let us put \( y = 1 \) in (3.6). We obtain
\[
\begin{align*}
-xf(1) &- \lambda f(x)f(1) \\
&= -\lambda f(0)f(1)(1 - mx) + [f(0) - xf(0) - \lambda f(x)f(0)](1 - n) \\
&+ \{\lambda[f(0)]^2 - f(0)\}(1 - m)(1 - n)
\end{align*}
\]
which can be written in the form
\[
\begin{align*}
\lambda f(x)[f(1) + (n - 1)f(0)] &+ \{m\lambda f(0) + 1\}[f(1) + (n - 1)f(0)] \\
-f(0)(mn - m)x - \lambda f(0)[f(1) + (n - 1)f(0)] &+ 0. \tag{3.18}
\end{align*}
\]

Now let us choose \( x_1 = 0, \ x_2 = \ldots = x_m = 0 \) in (3.5). We obtain
\[
(mn - m)f(0) = [\lambda f(1) + (m - 1)f(0)] + 1[f(1) + (n - 1)f(0)]. \tag{3.19}
\]
From (3.18) and (3.19), it is easy to derive the equation
\[ \lambda [f(1) + (n - 1)f(0)] [f(x) - (f(1) - f(0))x - f(0)] = 0. \] (3.20)
Since \( \lambda \neq 0 \) and \([f(1) + (n - 1)f(0)] \neq 0\), (3.4) follows immediately from (3.20).

Notice that this alternative method of deriving (3.4) does not make use of Lemma 2.

References


*Received August, 2004*