Some generalizations of Rédei’s theorem

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Abstract

By the famous theorems of Rédei, a set of \( q \) points in \( \text{AG}(2, q) \) (respectively \( p \) points in \( \text{AG}(2, p) \), \( p \) prime) is either a line or it determines at least \( \sqrt{q} + 1 \) (respectively \( \frac{p + 3}{2} \)) directions. We generalize these results on two fronts. First we provide bounds on the number of directions determined by a set of \( n \leq q \) points in a general projective plane of order \( q \). Secondly, given a dual \( n \)-arc in \( \Pi = \text{PG}(k, q) \) we consider \( \Pi \) as embedded in \( \Sigma = \text{PG}(k + 1, q) \) where \( E = \Sigma - \Pi \) is the associated affine space. A collection of affine points is a transversal set of \( K \) if any line incident with a \( k \)-fold point of \( K \) is incident with at most one point of \( S \). We reformulate Rédei’s results in the plane as results on transversal sets. In this setting we generalize Rédei’s theorems to higher dimensions. We also provide a new proof of a well known theorem on extending arcs in \( \text{PG}(k, q) \).

Keywords: Arc, dual arc, Rédei’s theorem.

1. Introduction

In 1970, the results of Rédei [12] provided the following Theorem.

Theorem 1.1 (Rédei’s Theorem). Let \( \pi = \text{PG}(2, q) \) with a distinguished line \( \ell_\infty \). Let \( S \) be a set of \( q \) points of \( \pi - \ell_\infty \) and let \( A \) be a collection of \( \delta \) points on \( \ell_\infty \) with the following property. Any line through a point of \( A \) intersects \( S \) in at most one point. If

\[
\delta > \begin{cases} 
\frac{q - 1}{2} & \text{if } q \text{ is prime} \\
q - \sqrt{q} & \text{otherwise}
\end{cases}
\]

then \( S \) is a subset of a line of \( \pi \).

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